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THE USE OF CHEBYSHEV POLYNOMIALS TO CONSTRUCT PERTURBED-MOTION TRAJECTORIES IN NON-LINEAR MECHANICS*

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Approximate solutions are constructed for a system of differential equations for perturbed motion over long time intervals. A Hilbert space projective method is used with a specially chosen metric to ensure uniform convergence. Analytic properties of Chebyshev polynomials together with simple operations enable one to derive a finite-dimensional system of projective algebraic equations.

1. Transformation of the problem. In the mechanics of perturbed motion one often has to solve a system of equations having the standard form

$$\mathbf{x}^{\cdot} = \mu\mathbf{X}(\mathbf{x}, t, \mu) \tag{1.1}$$

in the interval $t \in [0, T]$ with initial conditions $\mathbf{x}(0) = 0$. It is assumed that $\mathbf{x} \in B \subset \mathbf{R}^n$. The domain B corresponds to the domain of convergence of a power series for the vector function $\mathbf{X}$ in the vector variable $\mathbf{x}$. The function $\mathbf{X}$ is quasiperiodic in the argument t and can be represented in the form

$$\mathbf{X}(\mathbf{x}, t, \mu) = \sum_{\mathbf{k}\in \mathbf{Z}^m} \mathbf{X}_{\mathbf{k}}(\mathbf{x}, \mu) \exp(i\langle \boldsymbol{\omega}_0, \mathbf{k}\rangle t)$$

$$\langle \boldsymbol{\omega}_0, \mathbf{k}\rangle = \omega_{01}k_1 + \omega_{02}k_2 + \ldots + \omega_{0m}k_m$$

where $\mathbf{Z}$ is a set of integers. The vector coefficients are in turn represented in the form of multiple power series in the variables $x_1, \ldots, x_n$. We also assume that the vector function $\mathbf{X}$ is real-analytic in its arguments, including the small parameter μ.

The use of the initial condition $\mathbf{x}(0) = \mathbf{0}$ does not restrict the generality of the problem, because it can always be ensured by an appropriate change of coordinates.

It is assumed that the solution at a time T does not deviate too far from the origin of coordinates. In real calculations both the Fourier series and the power series are truncated, and the right-hand side $\mu\mathbf{X}$ is represented by the sum of a finite, though possibly large number of terms. The small parameter μ plays an important role. In particular, such a situation is found in the neighbourhood of an equilibrium position of a Hamiltonian system.

In this paper we propose the use of Chebyshev polynomials to construct solutions of problem (1.1). It is known that these polynomials enable one to obtain rapid convergence of uniform approximations to discontinuous functions /1/. We shall obtain uniform convergence as a consequence of convergence in a more complicated functional metric.

Consider Chebyshev polynomials of the first kind $T_n(\tau) = \cos(n \arccos \tau)$ $(n = 0, 1, \ldots)$. The functions $T_n(\tau)$ behave in an oscillatory manner in the interval $[-1, 1]$. However, the oscillations are concentrated towards the points $\tau = -1, 1$, while at the same time the right-hand sides of (1.1) oscillate quasiperiodically. In order to represent oscillatory motion in the interval $[0, T]$ with the help of Chebyshev polynomials in the interval $[-1, 1]$ it is necessary to perform a transformation of the independent coordinate t. So we put $\tau = \cos(\pi T^{-1}t)$.

If $t \in [0, T]$, then $\tau \in [-1, 1]$ and the relation is monotonic. The differentiation operator is expressed by the formula

$$\frac{d}{dt} = -\frac{\pi}{T}\sin\left(\frac{\pi}{T}t\right)\frac{d}{d\tau} = -\frac{\pi}{T}(1-\tau^2)^{1/2}\frac{d}{d\tau}$$

Using this we replace problem (1.1) with the problem

$$d\mathbf{x}/d\tau = \nu\mathbf{Y}(\mathbf{x}, \tau, \mu), \quad \mathbf{x}(1) = \mathbf{0}, \quad \tau \in [-1, 1], \ \nu = \mu T/\pi \tag{1.2}$$

where the new right-hand side has the form

$$\mathbf{Y}(\mathbf{x}, \tau, \mu) = -(1-\tau^2)^{-1/2}\mathbf{X}(\mathbf{x}, T\pi^{-1}\arccos\tau, \mu)$$

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It was remarked earlier that the vector function $\mathbf{X}$ can be represented in the form of a Poisson series

$$\mathbf{X}(\mathbf{x}, \tau, \mu) = \sum_{\mathbf{k}\in \mathbf{Z}^m} \sum_{\mathbf{l}\in \mathbf{Z}_+^n} \mathbf{X}_{\mathbf{lk}}(\mu)\, \mathbf{x}^{\mathbf{l}} \exp(i\langle \omega_0, \mathbf{k}\rangle t) \tag{1.3}$$

where $\mathbf{l} = (l_1, \ldots, l_n)$ is a multi-index, and the expression for the power monomial is $\mathbf{x}^{\mathbf{l}} = (x^1)^{l_1} (x^2)^{l_2} \ldots (x^n)^{l_n}$. The symbol $\mathbf{Z}_+$ denotes the set of non-negative integers.

In the expression for the function $\mathbf{Y}(\mathbf{x}, \tau, \mu)$ the quasiperiodic harmonics will be expressed by the functions $\cos(q_{\mathbf{k}} \arccos \tau)$ and $\sin(q_{\mathbf{k}} \arccos \tau)$, where $q_{\mathbf{k}} = T\pi^{-1}\langle \omega_0, \mathbf{k}\rangle$. From now on we shall use a right-hand side for (1.2) of the form

$$\mathbf{Y}(\mathbf{x}, \tau, \mu) = -\sum_{\mathbf{k}\in \mathbf{Z}^m} \sum_{\mathbf{l}\in \mathbf{Z}_+{}^n} \mathbf{x}^{\mathbf{l}} [\mathbf{A}_{\mathbf{lk}}(\mu)(1-\tau^2)^{-1/2}\cos(q_{\mathbf{k}}\arccos\tau) + \mathbf{B}_{\mathbf{lk}}(\mu)(1-\tau^2)^{-1/2}\sin(q_{\mathbf{k}}\arccos\tau)] \tag{1.4}$$

$$\mathbf{A}_{\mathbf{lk}}(\mu) = \operatorname{Re}\mathbf{X}_{\mathbf{lk}}(\mu), \quad \mathbf{B}_{\mathbf{lk}}(\mu) = -\operatorname{Im}\mathbf{X}_{\mathbf{lk}}(\mu), \quad \mathbf{X}_{\mathbf{l},-\mathbf{k}}(\mu) = \mathbf{X}^*_{\mathbf{lk}}(\mu)$$

(where the star denotes complex conjugation).

2. Description of the projective method. We will assume that the solution of problem (1.2) is defined for all $\tau \in [-1, 1]$ with $\mathbf{x}(\tau) \in Q$, where Q is the domain of convergence of the series for $\mathbf{X}$ in the variable $\mathbf{x} \in Q \subset \mathbf{R}^n$. It is known that problem (1.2) is equivalent to the equation

$$\mathbf{x} = \mathbf{Z}(\mathbf{x}, \mu), [\mathbf{Z}(\mathbf{x}, \mu)](\tau) = \int_1^\tau \nu \mathbf{Y}[\mathbf{x}(\alpha), \alpha, \mu]\, d\alpha \tag{2.1}$$

In order to construct a projective method for solving Eq.(2.1) it is necessary to specify an approximate functional space and to determine a system of projection operators in this space onto its finite-dimensional subspaces.

It is known that the Chebyshev polynomials of the first and second kinds form a complete orthogonal family of functions in the interval $[-1, 1]$ endowed with measures $d\mu_1 = (1-\tau^2)^{-1/2}d\tau$ and $d\mu_2 = (1-\tau^2)^{1/2}d\tau$ respectively.

Consider a Hilbert space of Sobolev class $H^1([-1, 1], \mu_1, \mu_2; \mathbf{R}^n)$ (henceforth simply H^1) with scalar product

$$(\mathbf{x}_1, \mathbf{x}_2)^1 = \int_{-1}^{1} (\mathbf{x}_1(\tau), \mathbf{x}_2(\tau))\, d\mu_1 + \int_{-1}^{1} (\mathbf{x}_1'(\tau), \mathbf{x}_2'(\tau))\, d\mu_2$$

where $(\,,\,)$ is the scalar product on $\mathbf{R}^n$ and the prime denotes differentiation with respect to τ. One can define H^1 as the completion of the set of continuously differentiable functions in $[-1, 1]$ in the norm

$$\|\mathbf{x}\|^1 = \left(\int_{-1}^{1} \|\mathbf{x}(\tau)\|^2 d\mu_1 + \int_{-1}^{1} \|\mathbf{x}'(\tau)\|^2 d\mu_2\right)^{1/2}$$

We will use the symbol $\mathrm{CA}([-1, 1], \mathbf{R}^n)$ (from now on simply CA) to denote the space of absolutely continuous functions in the interval $[-1. 1]$. One can introduce a Banach structures on CA using the norm defined by the formula

$$\|\mathbf{x}\|_A = \|\mathbf{x}(1)\| + \operatorname{Var}([-1, 1], \mathbf{x})$$

Using well-known methods /2/ and properties of the weighting functions $p_1(\tau) = (1-\tau^2)^{-1/2}$, and $p_2(\tau) = (1-\tau^2)^{1/2}$ one can prove that the space H^1 is embedded in CA. This means that convergence in the metric of H^1 reduces to convergence in the metric of CA, and in particular, to uniform convergence.

We will restrict the space H^1 to the subspace $H_0{}^1 = \{\mathbf{x} \in H^1 : \mathbf{x}(1) = 0\}$. It follows from the form of the right-hand side of Eq.(2.1) that it will always belong to $H_0{}^1$ if the function $\mathbf{Y}$ is sufficiently smooth and summable. Finally, as a domain of definition for the operator we define the set

$$\Omega = \{\mathbf{x} \in H_0{}^1 : \mathbf{x}(\tau) \in Q\ \forall\tau \in [-1, 1]\}$$

One can prove that Ω is a domain in $H_0{}^1$.

In order to construct a Galerkin scheme it is necessary to define a system of finite-dimensional subspaces $E_k \subset H_0^1 (k = 0, 1, \ldots)$ that exhaust the space H_0^1. To do this we will first construct a basis for the space H^1.

Suppose $\{\chi_k\}_{k=0}^{\infty}$ is an orthonormal basis for the space $H^1([-1, 1], \mu_1, \mu_2; \mathbf{R})$ (from now on H^{11}). Then it is easy to verify that the system of vector functions $\{\mathbf{e}_j\chi_k\}$ $(j = 1, \ldots, n;\ k = 0, 1, \ldots)$ is an orthonormal basis in H^1. Here the $\{\mathbf{e}_j\}_{j=1}^{n}$ are an orthonormal basis in $\mathbf{R}^n$.

The problem of constructing a basis in H^1 reduces to constructing a basis of scalar functions in H^{11}. If a function $u \in H^{11}$, then we automatically have $u \in L_2([-1, 1], \mu_1)$ and $u' \in L_2([-1, 1], \mu_2)$. It is known that in these spaces one can construct the orthonormal bases $\{f_k\}_{k=0}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ respectively, with $f_0 = \pi^{-1/2}T_0$, $f_k = (2/\pi)^{1/2}T_k$ and $g_k = (2/\pi)^{1/2}U_k$ $(k = 1, 2, \ldots)$.

It is well-known that for $k > 0$ the relation $T_k'(\tau) = kU_{k-1}(\tau)$ holds for Chebyshev polynomials of the first and second kinds. Hence $f_k'(\tau) = kg_k(\tau)\ (k > 0)$ and $f_0'(\tau) = 0$. In the metric of H^{11} the functions $\{f_k\}_{k=0}^{\infty}$ form an orthogonal system. If a normalization is performed then the system of functions $\{\chi_k\}_{k=0}^{\infty}$ such that

$$\chi_0(\tau) = \pi^{-1/2}T_0(\tau),\ \chi_k(\tau) = \{2/[\pi(1 + k^2)]\}^{1/2}T_k(\tau) \quad (k > 0)$$

is orthonormal in H^{11}.

It turns out that the system $\{\chi_k\}_{k=0}^{\infty}$ is complete in H^{11}, i.e. it is basis. From this it follows that if we know the expansion of the derivative $u'(\tau)$ of the function $u(\tau)$ in the space H^{11} with respect to the basis $\{g_k\}_{k=1}^{\infty}$

$$u'(\tau) = \sum_{k=1}^{\infty} v_k g_k(\tau)$$

then one can immediately write down the expansion

$$u(\tau) = \sum_{k=0}^{\infty} u_k f_k(\tau)$$

having put $u_k = v_k/k$ for $k > 0$. The coefficient u_0 is found from the condition $u(1) = 0$:

$$u_0 = -\sum_{k=1}^{\infty} k^{-1} v_k f_k(1)$$

We will now define the projection operators $P_k\ (k = 0, 1, \ldots)$ in H_0^1. To do this we represent them in the form $P_k = PP_k'$, where P_k' is the orthogonal projection operator in the space H^1 onto the finite dimensional subspace formed by the span of the first $n(k + 1)$ basis vectors $\mathbf{e}_1\chi_0, \mathbf{e}_2\chi_0, \ldots, \mathbf{e}_n\chi_k$. After projection $P_k'H_0^1$ can leave the space H_0^1. Therefore, in order to obtain a result in H_0^1, it is necessary to project the function $P_k'\mathbf{x}$ down to H_0^1, no longer orthogonally to H_0^1, but along the linear span of the vectors $\{\mathbf{e}_1\chi_0, \mathbf{e}_2\chi_0, \ldots, \mathbf{e}_n\chi_0\}$ so that $(PP_k'\mathbf{x})(1) = 0$. The operator P is given by the formula

$$P\mathbf{x} = \mathbf{x} - \mathbf{x}(1)\pi^{1/2}\chi_0 \quad (\mathbf{x} \in H^1)$$

Because the space $P_k'H^1$ exhausts all H^1, the spaces $E_k = P_k'H^1 \cap H_0^1$ exhaust all H_0^1. One can verify that $E_k = P_kH_0^1 = P_kH^1$. The operators P and P_k' are bounded. The Galerkin equations have the form

$$\mathbf{x}_k = P_k\mathbf{Z}(\mathbf{x}_k, \mu) \quad (\mathbf{x}_k \in E_kH_0^1,\ k > 0) \tag{2.2}$$

Further analysis is similar to that performed in /3/, using some results from /4/. First we verify the continuous differentiability of the operator $\mathbf{Z}$ in Ω. A sufficient condition for this is of the form $\mathbf{Y} \in L_2([-1, 1], \mu_2; C^1(Q))$, or in more detail

$$\|\mathbf{Y}\|_2^1 < +\infty \tag{2.3}$$

$$\|\mathbf{Y}\|_2^1 = \left(\int_{-1}^{1} \sup_{\mathbf{x} \in Q} (\|\mathbf{Y}(\mathbf{x}, \tau, \mu)\|, \ \|\mathbf{Y}_{\mathbf{x}}^{\cdot}(\mathbf{x}, \tau, \mu)\|)^2 d\mu_2\right)^{1/2}$$

If the function $\mathbf{X}$ is analytic in the domain of definition, then condition (2.3) holds uniformly for all initial conditions in some smaller domain, and the quantity $\|\mathbf{Y}\|_2^1$ is of order unity.

It follows from the boundedness of the linear operator P_k that the operator $P_k\mathbf{Z}$ is continuously differentiable. Furthermore, using the inequality

$$\|\mathbf{Z}\| \leqslant 2^{1/2}(1 + \pi^2)^{1/2}\|\nu\mathbf{Y}\|_2^1$$

we establish the continuous invertibility of the operator $I - \mathbf{Z}^{\cdot}(\mathbf{x}, \mu)\ (\mathbf{x} \in \Omega)$. This, of course, occurs if $\|\mathbf{Z}^{\cdot}(\mathbf{x}, \mu)\| < 1$, or

$$(\mu T/\pi)\,\|\,\mathbf{Y}\,\|_2{}^1 < [2(1+\pi^2)]^{-1/2} \tag{2.4}$$

From this we conclude that the length of the time interval over which convergence of the approximate to the exact solution is preserved must satisfy the condition $T < C\mu^{-1}$ where $C = O(1) = \pi\,[2(1+\pi^2)]^{-1/2}\,(\|\,\mathbf{Y}\,\|_2{}^1)^{-1}$. For example, one can put $T = C\,(2\mu)^{-1}$.

By analogy with the results of /3/, we obtain the assertion.

Theorem. Eq.(2.1) has a unique solution $\mathbf{x} \in H_0{}^1$. Furthermore, when condition (2.4) is satisfied, there exist an integer N and a real $\varepsilon > 0$ such that for any $k > N$ Eq.(2.2) has a unique solution $\mathbf{x}_k$ in the sphere $\|\,\mathbf{y} - \mathbf{x}\,\|^1 \leqslant \varepsilon$, and the convergence estimate

$$\|\,\mathbf{x}_k - \mathbf{x}\,\|^1 \leqslant \|\,\mathbf{x} - P_k\mathbf{x}\,\|^1 + \|\,\mathbf{x}_k - P_k\mathbf{x}\,\|^1 \to 0 \qquad (k \to \infty)$$

holds together with the two-sided estimate

$$c_1\,\|\,P_k\mathbf{Z}(\mathbf{x}) - P_k\mathbf{Z}(P_k\mathbf{x})\,\|^1 \leqslant \|\,\mathbf{x}_k - P_k\mathbf{x}\,\|^1 \leqslant c_2\,\|\,P_k\mathbf{Z}(\mathbf{x}) - P_k\mathbf{Z}(P_k\mathbf{x})\,\|^1$$

for some $c_1, c_2 > 0$.

This theorem guarantees the existence of Galerkin approximations and their uniform convergence to the exact solution. The solution of the finite-dimensional Eq.(2.2) can be found by an iterative method if the condition $\|\,P_k\mathbf{Z}^{\boldsymbol{\cdot}}(\mathbf{x})\,\| < 1$ holds. Because $\|\,P_k\,\| \leqslant 2^{1/2}\,(1+2\pi)^{1/2}$, the condition for the convergence of the iterative process takes the form

$$2\,[(1+2\pi)(1+\pi^2)]^{1/2}\nu\,\|\,\mathbf{Y}\,\|_2{}^1 < 1 \tag{2.5}$$

As an initial approximation one can take the function $\mathbf{x}_k = 0$, which corresponds to the solution of the unperturbed problem.

3. *Analytic techniques.* We will now consider the practical implementation of this solution method for Eq.(2.2). Suppose that k is sufficiently large and that the vector function $\mathbf{Y}(\mathbf{x}, \tau, \mu)$ has the form (1.4). The properties of Chebyshev polynomials give us the hope that k will not be too large.

In real calculations it is necessary to limit the number of terms in the expansion (1.4), i.e. the multi-index $\|\,\mathbf{k}\,\| \leqslant K$ $(\|\,\mathbf{k}\,\| = \max\,|\,k_i\,|\ (i = 1, \ldots, m))$, which corresponds to taking account of a finite (although perhaps large) number of harmonics in the Fourier series of the vector function $\mathbf{X}(\mathbf{x}, t, \mu)$. In turn the power series expansions of the coefficients of this series also have to be limited to a finite number of terms, i.e. the multi-index $\|\,\mathbf{l}\,\| \leqslant L$ $(\|\,\mathbf{l}\,\| = \max\,|\,l_i\,|\ (i = 1, \ldots, n))$, where the quantity L may depend on the multi-index $\mathbf{k}$. One usually takes into account those terms from series (1.4) which have less than a specified order of smallness in the parameter μ. Thus instead of the function $\mathbf{Y}(\mathbf{x}, \tau, \mu)$ we shall consider

$$\mathbf{Y}^*(\mathbf{x}, \tau, \mu) = -\sum_{\|\mathbf{k}\|\leqslant K}\sum_{\|\mathbf{l}\|\leqslant L}\mathbf{x}^{\mathbf{l}}\,[\mathbf{A}_{\mathbf{lk}}(\mu)\,(1-\tau^2)^{-1/2}\cos(q_{\mathbf{k}}\arccos\tau) + \mathbf{B}_{\mathbf{lk}}(\mu)\,(1-\tau^2)^{-1/2}\sin(q_{\mathbf{k}}\arccos\tau)] \tag{3.1}$$

Our aim is to find an expansion of the function $\mathbf{Y}^*$ in terms of Chebyshev polynomials of the second kind, using a representation of the vector $\mathbf{x}(\tau)$ in the form of a sum of the first k Chebyshev polynomials of the first kind. Such a representation of $\mathbf{x}(\tau)$ corresponds to the expansion of $\mathbf{Y}^*(\mathbf{x}(\tau), \tau, \mu)$ in the space $L_2([-1, 1], \mu_2; \mathbf{R}^n)$ with respect to the basis $\{\mathbf{e}_i g_j\}_{i=1,\ldots,n}^{j=1,\ldots}$. After this we automatically obtain the expansion of the primitive of $\mathbf{Y}^*(\mathbf{x}(\tau), \tau, \mu)$ in Chebyshev polynomials of the first kind, which corresponds to its expansion in the space $L_2([-1, 1], \mu_1; \mathbf{R}^n)$ with respect to the basis $\{\mathbf{e}_i f_j\}_{i=1,\ldots,n}^{j=0,1,\ldots}$ or in the space H^1 with respect to the basis $\{\mathbf{e}_i \chi_j\}_{i=1,\ldots,n}^{j=0,1,\ldots}$.

We will give formulae necessary for achieving this aim. The first is the expansion

$$\frac{\cos(q\arccos\tau)}{(1-\tau^2)^{1/2}} = \sum_{l=0}^{\infty} c_l U_l(\tau), \qquad \frac{\sin(q\arccos\tau)}{(1-\tau^2)^{1/2}} = \sum_{l=0}^{\infty} s_l U_l(\tau)$$

Formulae for computing the coefficients

$$c_l = \frac{2}{\pi}\,\frac{(l+1)\,[1+(-1)^l\cos q\pi]}{(l+1)^2 - q^2}, \quad s_l = \frac{2}{\pi}\,\frac{(l+1)\,(-1)^l\sin q\pi}{(l+1)^2 - q^2} \tag{3.2}$$

cover the case of integer parameter q, which in the particular case $q = 0$ indicates the resonance of the corresponding harmonics in (1.3), and in these cases we have the simpler formulae

$$c_{2l} = \frac{4}{\pi}\frac{2l+1}{(2l+1)^2-q^2}, \quad c_{2l+1} = 0 \quad (l = 0, 1, \ldots; \; q = 2r; \; r \in \mathbf{Z})$$
$$c_{2l} = 0, \quad c_{2l+1} = \frac{4}{\pi}\frac{2l+2}{(2l+2)^2-q^2} \quad (l = 0, 1, \ldots; \; q = 2r+1; \; r \in \mathbf{Z})$$
$$s_r = 1, \; s_l = 0 \quad (l = 0, 1, \ldots; \; l \neq r; \; r \in \mathbf{Z}_+)$$

which are obtained from (3.2) by taking the limits $q \to 2r$, $q \to 2r+1$ and $q \to r+1$, respectively.

Assuming that $\mathbf{x}_k \in E_k = P_k H_0^1$ we obtain the representation

$$\mathbf{x}_k(\tau) = \sum_{l=0}^{k} \mathbf{x}_{kl} T_l(\tau), \quad \mathbf{x}_{kl} \in \mathbf{R}^n, \; \mathbf{x}_k(1) = 0$$

The vector $\mathbf{x}_{kl}$ can be represented in the form of a column of its coordinates $\mathbf{x}_{kl} = (x_{kl}^1, x_{kl}^2, \ldots, x_{kl}^n)^T$. Hence taking into account the condition $\mathbf{x}_k(1) = 0$ there should be kn unknown scalar quantities in the Galerkin Eq.(2.2). In the coordinate representation

$$x_k^r(\tau) = \sum_{l=0}^{k} x_{kl}^r T_l(\tau) \quad (r = 1, \ldots, n)$$

Because the power multiplier in (3.1) has the form $\mathbf{x}^l = (x^1)^{l_1}(x^2)^{l_2}\ldots(x^n)^{l_n}$, one uses the well-known expansion of each individual factor in terms of Chebyshev polynomials of the first kind to find a similar representation for the entire product. To do this we use the well-known identity

$$T_m(\tau) T_n(\tau) = 2^{-1}(T_{m-n}(\tau) + T_{m+n}(\tau)) \; (m, n \in \mathbf{Z}) \tag{3.3}$$

Using (3.3) one can obtain the expansion of the entire monomial $\mathbf{x}^l$ in terms of Chebyshev polynomials of the first kind. Finally, in order to expand all the terms of (3.1) in terms of Chebyshev polynomials of the second kind we must use the identity

$$U_m(\tau) T_n(\tau) = 2^{-1}(U_{m-n}(\tau) + U_{m+n}(\tau)) \quad (m, n \in \mathbf{Z})$$

Finally, after all the substitutions and multiplications, we bring together similar terms in (3.1) and neglect all terms which contain $U_l(\tau)$ with $l > k-1$. This also enables us to exclude unnecessary terms at earlier stages of the computation, during the multiplications. Denoting the truncation of a series by square brackets, we obtain

$$[\mathbf{Y}^*(\mathbf{x}_k(\tau), \tau, \mu)]_{k-1} = \sum_{l=0}^{k-1} \mathbf{y}_l(x_{k1}, \ldots, x_{kk}, \mu) U_l(\tau) \tag{3.4}$$

The vector coefficient $\mathbf{y}_l$ depends polynomially on the unknown quantities x_{kl}^r $(l = 1, \ldots, k; \; r = 1, \ldots, n)$. The initial condition $\mathbf{x}_k(1) = 0$ ensures the relation

$$x_k^r(1) = -\sum_{l=1}^{k} x_{kl}^r T_l(1) = -\sum_{l=1}^{k} x_{kl}^r \quad (r = 1, \ldots, n)$$

which should also be used in deriving expansion (3.4). The truncation operation on a Chebyshev series corresponds to the action of the orthogonal projection operator P_k' (in the space of derivatives $L_2([-1, 1], \mu_2; \mathbf{R}^n)$). Computing the primitive and applying the projection operator P_k we have

$$[P_k \mathbf{Z}(x_k, \mu)](\tau) = \int_1^{\tau} \nu [\mathbf{Y}^*[x(\alpha), \alpha, \mu]]_{k-1} d\alpha =$$
$$\left(-\sum_{l=1}^{k} \nu l^{-1} \mathbf{y}_{l-1}\right) T_0(\tau) + \sum_{l=1}^{k} \nu l^{-1} \mathbf{y}_{l-1} T_l(\tau)$$

We obtain as a result a system of kn algebraic equations, corresponding to the functional Eq.(2.2):

$$x_{kl}^r = \nu l^{-1} y_{l-1}^r(x_{k1}, \ldots, x_{kk}, \mu) \; (r = 1, \ldots, n; \; l = 1, \ldots, k)$$

This system can be solved by iteration.

REFERENCES

1. PASHKOVSKII S., Numerical Applications of Chebyshev Polynomials and Series, Nauka, Moscow, 1983.
2. SCHWARZ L., Analysis, 1, Mir, Moscow, 1972.

3. KOSENKO I.I., On phase trajectory construction for Hamiltonian systems in the neighbourhoods of equilibrium positions, Prikl. Mat. Mekh., 53, 4, 1989.
4. KRASNOSEL'SKII M.A., VAINIKKO G.M., ZABREIKO P.P., RUTITSKII YA.B. and STETSENKO V.YA., Approximate Solutions of Operator Equations, Nauka, Moscow, 1969.

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RECURRENT ESTIMATION AND IDENTIFICATION OF THE PARAMETERS IN NON-LINEAR DETERMINISTIC SYSTEMS*

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Estimation of the phase states and parameters of non-linear deterministic systems of differential equations is reduced to the determination of initial data which minimize a certain functional which depends on observations and prior information. Equations are derived for an optimum non-linear filter whose realization demands repeated integration of auxiliary systems of differential equations. A modified, simpler filter, which is nearly optimum in many quite typical situations, is constructed. Consideration is given to the problem of estimation based on partly-known initial data, a special case of which is identifying the parameters of a system whose phase states are known at the initial time. In the linear case, if there is no a priori information, the results obtained here represent a deterministic version of Kalman filtering. The most constructive results in estimation have been obtained for linear systems (for general approaches see /1/, for recurrent filtration given known a priori information of a statistical nature about the initial data and noise in the object and in the observations, see /2/, for a deterministic version of recurrent estimation along game-theoretic lines, assuming known restrictions on noise, see /3/, and for a deterministic version of Kalman filtering see /4, 5/).

1. ***Statement of the problem.*** We shall consider questions relating to the estimation of non-linear systems of ordinary differential equations

$$X' = f(s, X),\ s \geqslant t_0 \tag{1.1}$$

with observations

$$y(s) \doteq \varphi(s, X(s)),\quad s \geqslant t_0 \tag{1.2}$$

The prime denotes differentiation with respect to s, X, y are column vectors with n and m components, respectively, the approximate equality in (1.2) indicates that the observations involve an unknown degree of noise.

The identification of a parameter Λ (where Λ is an l-vector) in the system

$$X' = f(s, X, \Lambda) \tag{1.3}$$

given observations (1.2) and taking into account the relations

$$\Lambda' = 0 \tag{1.4}$$

obviously reduces to estimating the phase variables in system (1.3), (1.4) given observations (1.2) (the function φ in (1.2) may then depend on the parameter Λ: $\varphi = \varphi(s, X(s), \Lambda)$).

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